

AB. The outer flat copper case.

C. Plate of amalgamated lead held in grooves in the cap D and the slip E.

F shows the hole in the cap through which the solution is introduced, and by the introduction of a glass tube through this hole the state of the charge is seen by observing the colour; the interior surface of the case forms the negative, and the amalgamated lead the positive electrode.

January 19, 1882.

THE PRESIDENT in the Chair.

The Presents received were laid on the table and thanks ordered for them.

The following Papers were read :—

I. "On certain Definite Integrals." No. 10. By W. H. L. RUSSELL, F.R.S. Received December 31, 1881.

Let $a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 = \alpha + \beta(\mu + \nu x + \rho x^2) + \gamma(\mu + \nu x + \rho x^2)^2$.

Then we find the condition

$$8a_1a_4^2 + a_3^3 - 4a_2a_3a_4 = 0,$$

together with the equations—

$$\gamma\rho^3 = a_4, \quad 2a_4\nu = \rho a_3, \quad \beta\nu + 2\gamma\mu\nu = a_1,$$

when there are three equations connecting the five quantities β , γ , μ , ν , ρ .

Under these conditions we shall have—

$$\int_b^a dx \epsilon^{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4} = \int_{\mu + \nu b + \rho b^2}^{\mu + \nu a + \rho a^2} \frac{dz \cdot \epsilon^{\alpha + \beta z + \gamma z^2}}{\sqrt{4\rho(z - \mu) + \nu^2}} \cdot \cdot \cdot \quad (220).$$

In the same way, if

$$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \\ = \alpha + \beta(\mu + \nu x + \rho x^2) + \gamma(\mu + \nu x + \rho x^2)^2 + \delta(\mu + \nu x + \rho x^2)^3,$$

we have similarly—

$$\int_b^a dx \epsilon^{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6} = \int_{\mu + \nu b + \rho b^2}^{\mu + \nu a + \rho a^2} \frac{dz \cdot \epsilon^{\alpha + \beta z + \gamma z^2 + \delta z^3}}{\sqrt{4\rho(z - \mu) + \nu^2}} \cdot \cdot \cdot \quad (221).$$

There will be six equations connecting the quantities a_1 , a_2 , a_3 , a_4 ,

a_5, a_6 , with $\beta, \gamma, \delta, \mu, \nu, \rho$; I have not observed that these equations lead to an equation of condition between $a_1, a_2, \dots$ *

This transformation will apply equally well to all integrals of the form—

$$\int dx \phi(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4) \dots \dots \dots (222),$$

$$\text{and} \quad \int dx \phi(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6) \dots \dots (223).$$

If we consider the integrals

$$\int_0^a d\theta \sin(a + b\theta + c\theta^2 + \dots \alpha\theta^n) \dots \dots \dots (224),$$

$$\text{and} \quad \int_0^a d\theta \cos(a + b\theta + c\theta^2 + \dots \beta\theta^n) \dots \dots \dots (225),$$

where α is not infinite, as it is in Fresnel's integrals, we must proceed by expansion.

$$\text{Let} \quad \Theta = a + b\theta \dots e\theta^n, \quad u = \sin \Theta, \quad v = \cos \Theta,$$

$$\frac{du}{d\theta} = \frac{d\Theta}{d\theta} \cdot v, \quad \frac{dv}{d\theta} = -\frac{d\Theta}{d\theta} \cdot u,$$

$$\frac{d^2 u}{d\theta^2} = \frac{d^2 \Theta}{d\theta^2} v + \left(\frac{d\Theta}{d\theta} \right) u, \quad \frac{d^2 v}{d\theta^2} = -\frac{d^2 \Theta}{d\theta^2} u - \frac{d\Theta^2}{d\theta^2} v.$$

Proceeding in this manner, and remembering that when $\theta=0$, $\sin \Theta = \sin a$, $\cos \Theta = \cos a$, we are able to develop u and v with great facility, and so obtain the integrals.

The following method may be very generally applied. Let

$$\phi(x) = A_0 + A_1 \left(\frac{x-a}{x+a} \right) + A_2 \left(\frac{x-a}{x+a} \right)^2 + \dots$$

where (a) is an arbitrary quantity. Then we shall have—

$$\begin{aligned} \int dx \phi(x) &= \int \frac{dx}{(x+a)^2} (x+a)^2 \phi(x) \\ &= 4a^2 \int \frac{dx}{(x+a)^2} \left\{ 1 + 2 \left(\frac{x-a}{x+a} \right) + 3 \left(\frac{x-a}{x+a} \right)^2 + \dots \right\} \\ &\quad \left\{ A_0 + A_1 \frac{x-a}{x+a} + A_2 \left(\frac{x-a}{x+a} \right)^2 + A_3 \left(\frac{x-a}{x+a} \right)^3 + \dots \right\} \\ &= 2a \left\{ B_0 \cdot \frac{x-a}{x+a} + \frac{B_1}{2} \left(\frac{x-a}{x+a} \right)^2 + \frac{B_2}{3} \left(\frac{x-a}{x+a} \right)^3 + \dots \right\} \dots \dots (226), \end{aligned}$$

where $B_0 = A_1$, $B_1 = A_1 + 2A_0$, $B_2 = A_2 + 2A_1 + 3A_0 \dots$

The calculation of $B_0, B_1 \dots$ is very easy. Form by addition the

* Jan. 23. But Mr. Spottiswoode has discovered two conditions, which will, I hope, be inserted in the next paper on the subject.

series $A_0, A_0 + A_1, A_0 + A_1 + A_2 + \dots$. Then by a second addition we have $A_0, A_1 + 2A_0, A_2 + 2A_1 + 3A_0$, or $B_0, B_1, B_2, \dots$.

Moreover, if the series $A_0, A_1, A_2, \&c.$, is convergent, the series $B, B_1, B_2, \&c.$, is also convergent, because

$$\frac{B_{n+1}}{B_n} = \frac{A_{n+1} + 2A_n + 3A_{n-1} + 4A_{n-2} + \dots}{A_n + 2A_{n-1} + 3A_{n-2} + 4A_{n-3} + \dots},$$

and if A_{n+1} is in the limit less than A_n , A_n less than $A_{n-1} \dots B_{n+1}$ must in the limit be less than B_n .

Hence if the limits are greater than a , and the difference between either of them and a small when compared with (a) , we shall have in many cases excellent convergence.

This method of converging to the value of definite integrals is useful in dynamical problems. Another is as follows.

Let
$$\phi(x) = a + bx + cx^2 + ex^3 + \dots + rx^n = z^2,$$

then

$$\phi'(x) \frac{dx}{dz} = 2z,$$

$$\phi''(x) \frac{dx^2}{dz^2} + \phi'(x) \frac{d^2x}{dz^2} = 2,$$

$$\phi'''(x) \frac{dx^3}{dz^3} + 3\phi''(x) \frac{dx}{dz} \cdot \frac{d^2x}{dz^2} + \phi'(x) \frac{d^3x}{dz^3} = 0, \&c.$$

When $z=0$, x is one of the roots of $a + bx + cx^2 + \dots + rx^n = 0$; call it (α) . Then

$$\begin{aligned} \frac{dx}{dz} &= 0, & \frac{d^2x}{dz^2} &= -\frac{2}{\phi'(\alpha)}, \\ \frac{d^3x}{dz^3} &= 0, & \frac{d^4x}{dz^4} &= -\frac{12\phi''(\alpha)}{(\phi'(\alpha))^3}. \end{aligned}$$

Hence

$$x = \alpha + \frac{z^2}{\phi'(\alpha)} - \frac{\phi''(\alpha)}{(\phi'(\alpha))^3} \cdot \frac{z^4}{2} + \dots,$$

$$\int x dz = \alpha z + \frac{z^3}{3\phi'(\alpha)} - \frac{\phi''(\alpha)}{(\phi'(\alpha))^3} \cdot \frac{z^5}{2 \cdot 5} + \dots$$

But

$$\int x dz = xz - \int z dx.$$

Hence
$$\int dx \sqrt{a + bx + cx^2 + ex^3 + \dots + ex^n}$$

$$= xz - \left\{ \alpha z + \frac{z^3}{3\phi'(\alpha)} - \frac{\phi''(\alpha)}{(\phi'(\alpha))^3} \cdot \frac{z^5}{2 \cdot 5} + \dots \right\} \quad (227).$$

The limits are arbitrary; but the series evidently requires that $\phi'(\alpha)$ should be considerable.

Since

$$\int_0^1 dx x^{n-1} = \frac{1}{n},$$

$$\int_0^1 dx(1-x)x^{n-1} = \frac{1}{n(n+1)},$$

$$\int_0^1 dx(1-x)^2 x^{n-1} = \frac{1 \cdot 2}{n(n+1)(n+2)},$$

$$\int_0^1 dx(1-x)^3 x^{n-1} = \frac{1 \cdot 2 \cdot 3}{n(n+1)(n+2)(n+3)},$$

&c. = &c., we have, if $\phi(\theta) = A_0 + A_1\theta + A_2\theta^2 + \dots$,

$$\int_0^1 \phi(1-x) dx x^{n-1} = \frac{A_0}{n} + \frac{A_1}{n(n+1)} + \frac{1 \cdot 2 \cdot A_2}{n(n+1)(n+2)} + \&c. \quad (228).$$

This formula is most useful when n is very large, or when (n) is very small. Suppose $n=100$, then the tenth term is less than

$$\frac{A_9}{60,000,000,000,000},$$

and after this the terms continue to diminish, though not so rapidly.

When n is very small, we shall come to an integer (r) which does not differ sensibly from $n+r$, and therefore we are able to write the integral:—

$$\begin{aligned} \int_0^1 \phi(1-x) dx x^{n-1} &= \frac{A_0}{n} + \frac{A_1}{n(n+1)} + \frac{1 \cdot 2 \cdot A_2}{n(n+1)(n+2)} + \dots \\ &\quad + \frac{1 \cdot 2 \cdot 3 \dots r}{n(n+1) \dots (n+r)} (A_r + A_{r+1} + A_{r+2} \dots) \\ &= \frac{A_0}{n} + \frac{A_1}{n(n+1)} + \frac{1 \cdot 2 \cdot A_2}{n(n+1)(n+2)} + \frac{1 \cdot 2 \cdot 3 \cdot A_3}{n(n+1)(n+2)(n+3)} + \dots \\ &\quad + \frac{1 \cdot 2 \cdot 3 \dots r}{n(n+1) \dots (n+r)} (\phi(0) - A_0 - A_1 \dots - A_{r-1}). \quad (229). \end{aligned}$$

In the same way we are able to find:—

$$\int_0^1 \phi(1-x) dx x^{n-1} (\log_{\epsilon} x)^2 \dots \dots \dots (230)$$

(since $\int_0^1 x^{n-1} (\log_{\epsilon} x)^2 = \frac{2}{(n+1)^3}$);

also $\int_0^1 \chi(1-x) dx x^{n-1} \phi(\log_{\epsilon} x) \dots \dots \dots (231).$

$$\int_0^1 dx x^{n-1} \phi(1-x, \log_{\epsilon} x) \dots \dots \dots (232).$$

We must, however, observe that the functions involving $\log x$ are supposed convergent, whatever the value of $\log x$, when expanded.

In a similar way we obtain

$$\int_0^\pi d\theta \sin \theta \phi(\sin^2 \theta) \cos^{2r} \theta \quad . \quad . \quad . \quad (233).$$

January 19, 1882.—From formula (226) it immediately follows that—

$$\int_\beta^\alpha d\phi(x) = 2\beta \left\{ B_0 \frac{\alpha-\beta}{\alpha+\beta} + \frac{B_1}{2} \left(\frac{\alpha-\beta}{\alpha+\beta} \right)^2 + \dots \right\}$$

II. "Manometric Observations in the Electric Arc." By Professor DEWAR, M.A., F.R.S. Received January 14, 1882.

The experiments recorded in my former paper, entitled "Studies on the Electric Arc,"* together with the numerous observations made conjointly with Professor Liveing on the spectrum of the arc discharge between carbon electrodes in different gases, led me to ascertain if the interior of the gaseous envelope of the ordinary arc showed any peculiarities of pressure. Pressure might be caused by the motion of the gas particles, the transit of material from pole to pole, electric action, or indirectly by chemical combinations taking place in the arc. As any effect due to the above causes must necessarily be very small, a delicate manometer capable of measuring easily the $\frac{1}{1000}$ th of a millimetre of water pressure had to be employed. The records of such an instrument in the present series of experiments, are complicated by the indirect action of the hot currents of air passing the poles, and the irregularities in the steadiness of the arc which undoubtedly cause marked variations of pressure; yet by multiplying and varying the conditions of the experiments, it is possible to eliminate these secondary effects and secure reliable results. The general appearance of the apparatus used is shown in the diagram. A and B are two hollow carbons, similar to those I formerly employed in the separation of cyanogen from the arc.† They must be free from all porosity before they can be used in the experiments to be detailed, and the drilled hole should not be less than 3 millims. in diameter. In order to fill up minute apertures in the carbons, and render them non-porous, they are placed in a porcelain tube and heated to a white heat in a current of coal-gas saturated with vapour of benzole. This treatment causes the deposition of a layer of dense metallic carbon over the surface of the tubes which renders them capable of withstanding a considerable interior pressure of air or other gas without exhibiting leakage. The hollow poles are connected by means of tubing

* "Proc. Roy. Soc.," vol. 30, p. 85.

† "Proc. Roy. Soc.," vol. 29, p. 188.